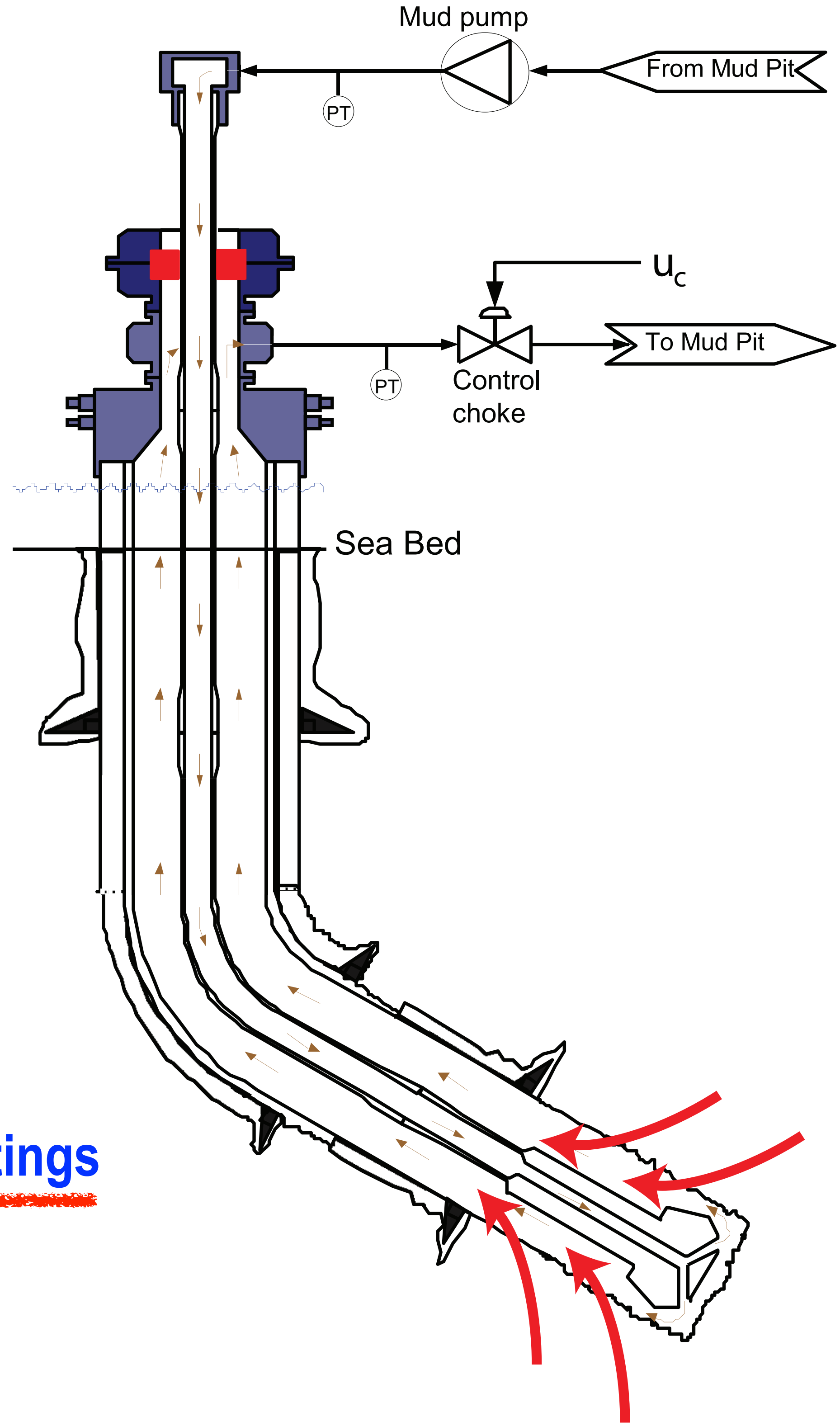
PDE Backstepping Control of Congested Traffic

Miroslav Krstic

Oil Drilling



Highway (single-lane) for gas, water, oil, mud, rock cuttings

Choke valve = ramp metering

Huan Yu

Control of Flow Dynamics in CONGESTED Traffic

Suppressing STOP-AND-GO oscillations **(coupled PDEs)**

Control of Flow Dynamics in CONGESTED Traffic

Suppressing STOP-AND-GO oscillations (coupled PDEs)

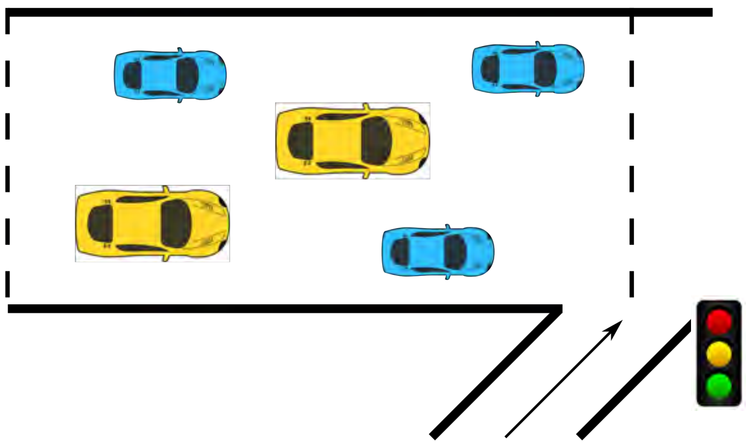
Keeping SHOCKS from drifting upstream (ODE with delays)

Control of Flow Dynamics in CONGESTED Traffic

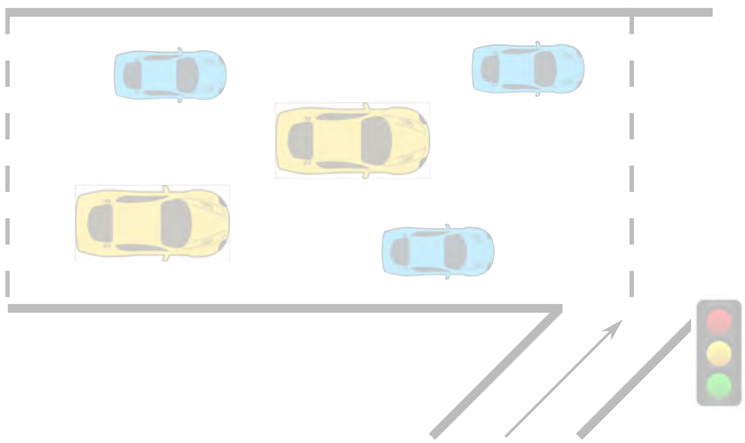
Suppressing STOP-AND-GO oscillations (coupled PDEs)

Keeping SHOCKS from drifting upstream (ODE with delays)

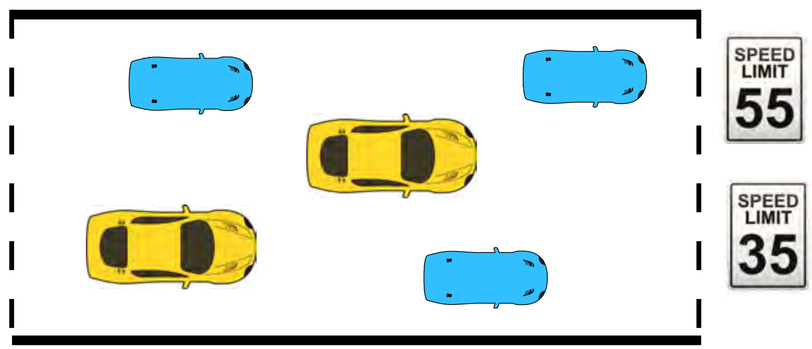
Maximizing flow through BOTTLENECK (extremum seeking)



Ramp metering, **flow rate actuated**



Ramp metering, flow rate actuated



Varying speed limit, velocity actuated

PDE Backstepping Control of
Stop-and-Go Traffic

Aw-Rascle-Zhang model

$\rho(x,t) =$ traffic density, $v(x,t) =$ traffic speed

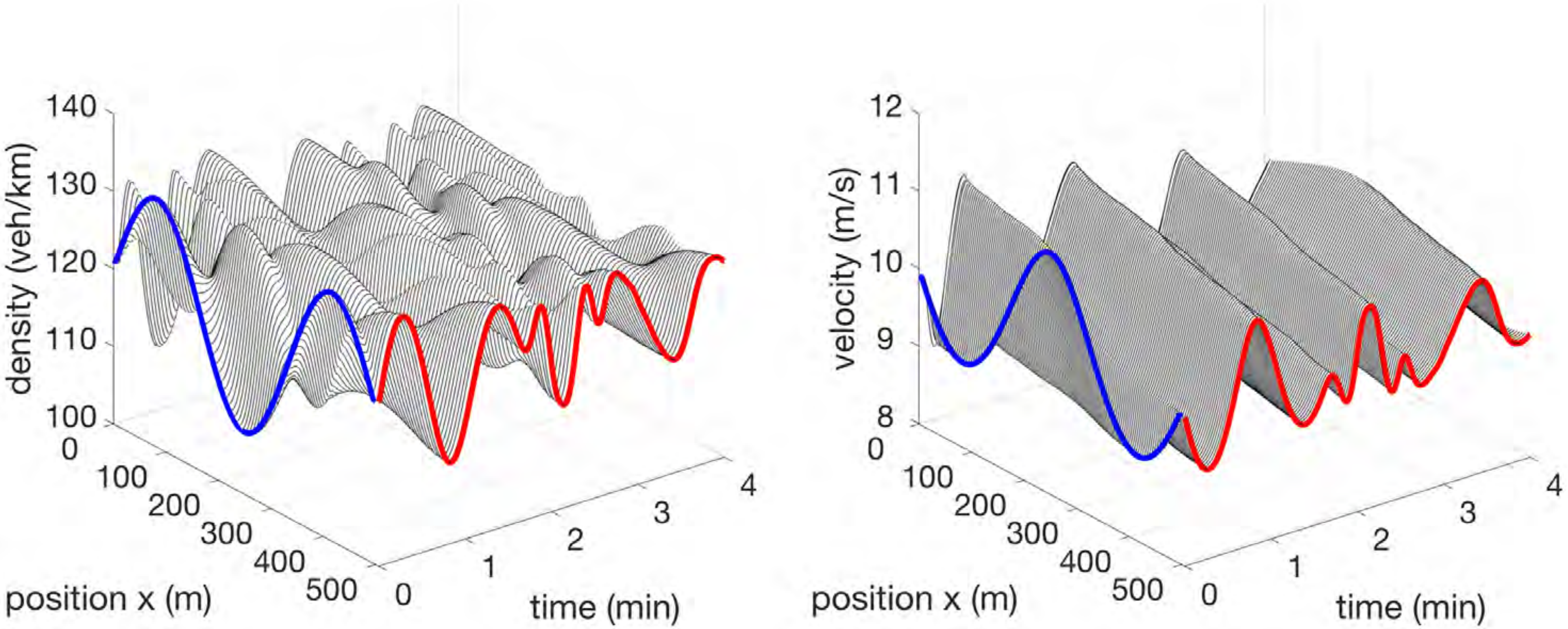
LWR model $\boxed{\partial_t \rho + \partial_x(\rho v) = 0}$

$$\partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = \frac{V(\rho)-v}{\tau}$$

second-order, **macroscopic**, **nonlinear hyperbolic PDEs**

Freeway traffic open-loop simulation

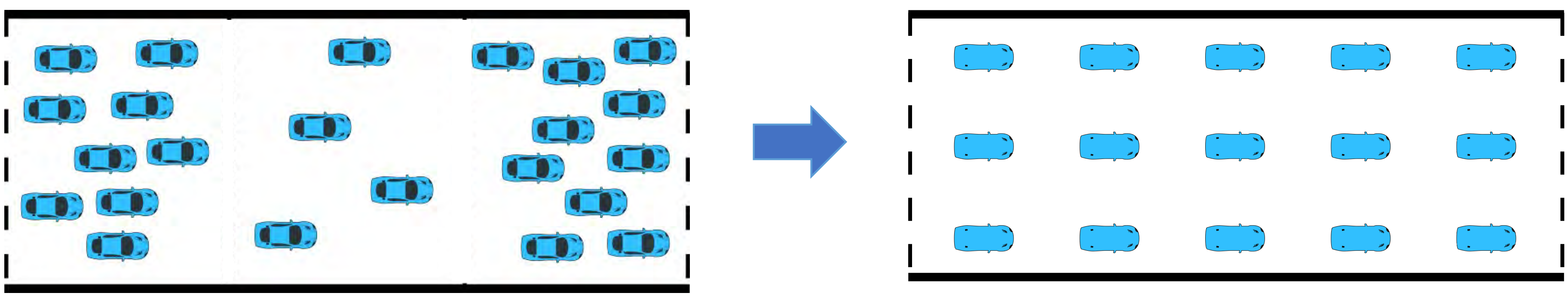
$\rho^\star = 120$ vehicles/km, $\quad v^\star =$ 22 mph $\quad L = 500$ m, $\quad \tau = 60$ s



eigenvalues in RHP!

Control objective

$\rho(x,t), v(x,t)$ $\rho^\star, v^\star$



Linearized ARZ model

$\tilde{\rho}(x,t) =$ density disturbance, $\tilde{v}(x,t) =$ speed disturbance

$$\begin{aligned} \tilde{\rho}_t + {\color{red}v^\star}\tilde{\rho}_x &= -\,\rho^\star\tilde{v}_x \\ \tilde{v}_t - {\color{blue}(\gamma p^\star - v^\star)}\tilde{v}_x &= -\,\frac{\tilde{v}+\tilde{p}}{\tau} \end{aligned}$$

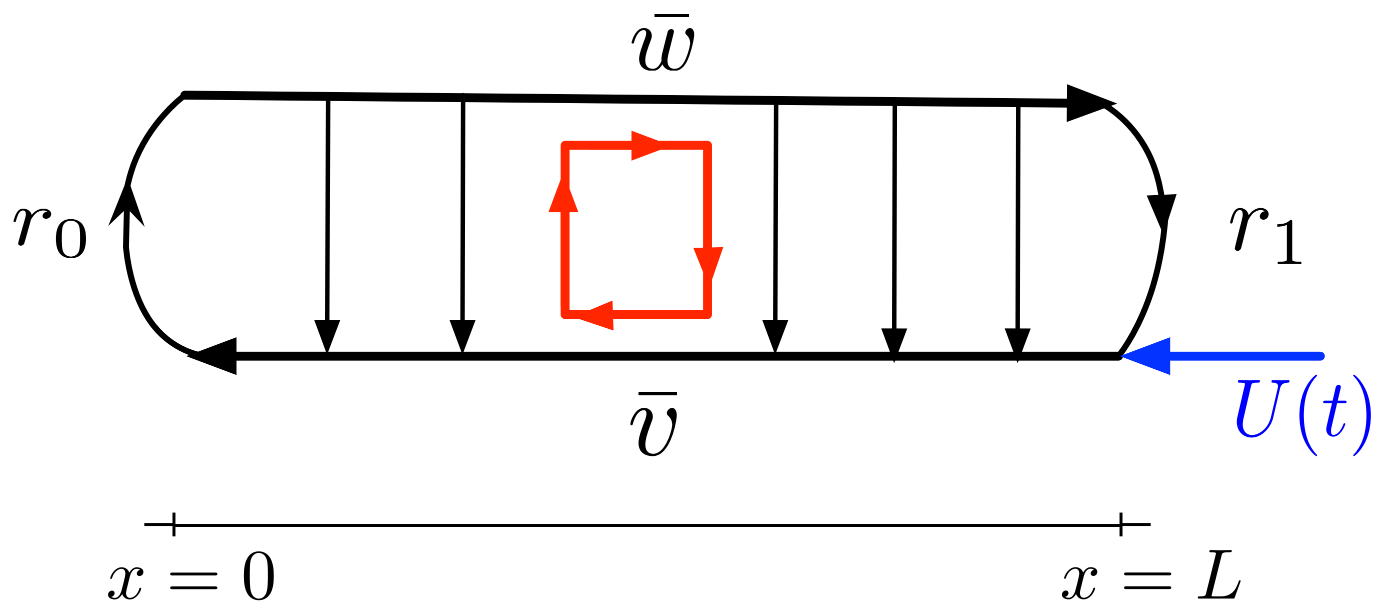
Linearized ARZ model

$\tilde{\rho}(x,t) =$ density disturbance, $\tilde{v}(x,t) =$ speed disturbance

$$\tilde{\rho}_t + v^\star \tilde{\rho}_x = -\rho^\star \tilde{v}_x$$
$$\tilde{v}_t - (\gamma p^\star - v^\star)\tilde{v}_x = -\frac{\tilde{v} + \tilde{p}}{\tau}$$

- density disturbance $\tilde{\rho}$ “propagates” at traffic speed setpoint $v^\star$
- speed disturbance $\tilde{v}$ “**counter**-propagates” at $\gamma p^\star - v^\star$

Coupled 2×2 hyperbolic PDE model



Positive feedback throughout the domain

green/red light
at downstream ramp

Theorem. *Full-state feedback control law*

$$\begin{aligned} U_{\mathrm{out}}(t) = & - r_0\rho^\star(v(L,t) - v^\star) \\ & + r_0\rho^\star \int_0^L \left[M(L-\xi) - r_0 K(L,\xi) \exp\left(\frac{\xi}{\tau v^\star}\right)\right] (v(\xi,t) - v^\star) d\xi \\ & + k_0 \int_0^L K(L,\xi) \exp\left(\frac{\xi}{\tau v^\star}\right) (\rho(\xi,t)v(\xi,t) - \rho^\star v^\star) d\xi, \end{aligned}$$

makes equilibrium $\bar{w} \equiv \bar{v} \equiv 0$ *exponentially stable in* L_2 *and equilibrium reached in finite time* $t = t_f$.

Kernel equations

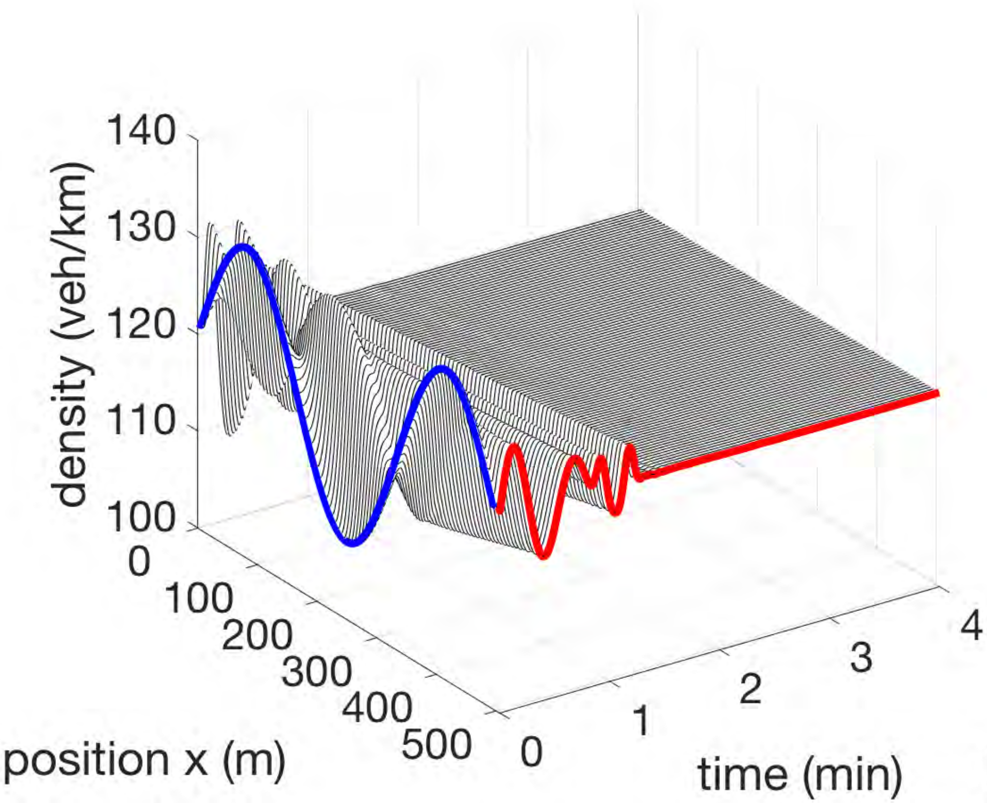
$$(\gamma p^\star - v^\star) K_x - v^\star K_\xi = c(\xi) K(x-\xi, 0)$$

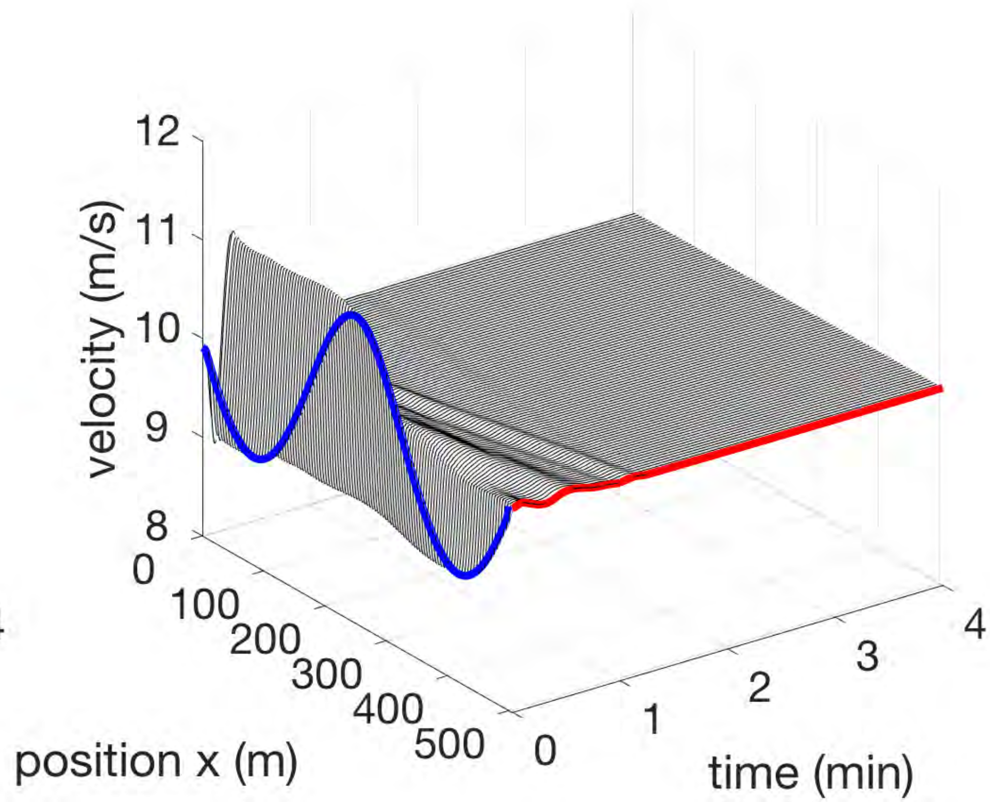
$$K(x,x) = -\frac{c(x)}{\gamma p^\star}$$

where $\{0 \leq \xi \leq x \leq L\}$

$$M(x) = -K(x,0)$$

Full-state feedback simulation





Boundary observer for traffic estimation

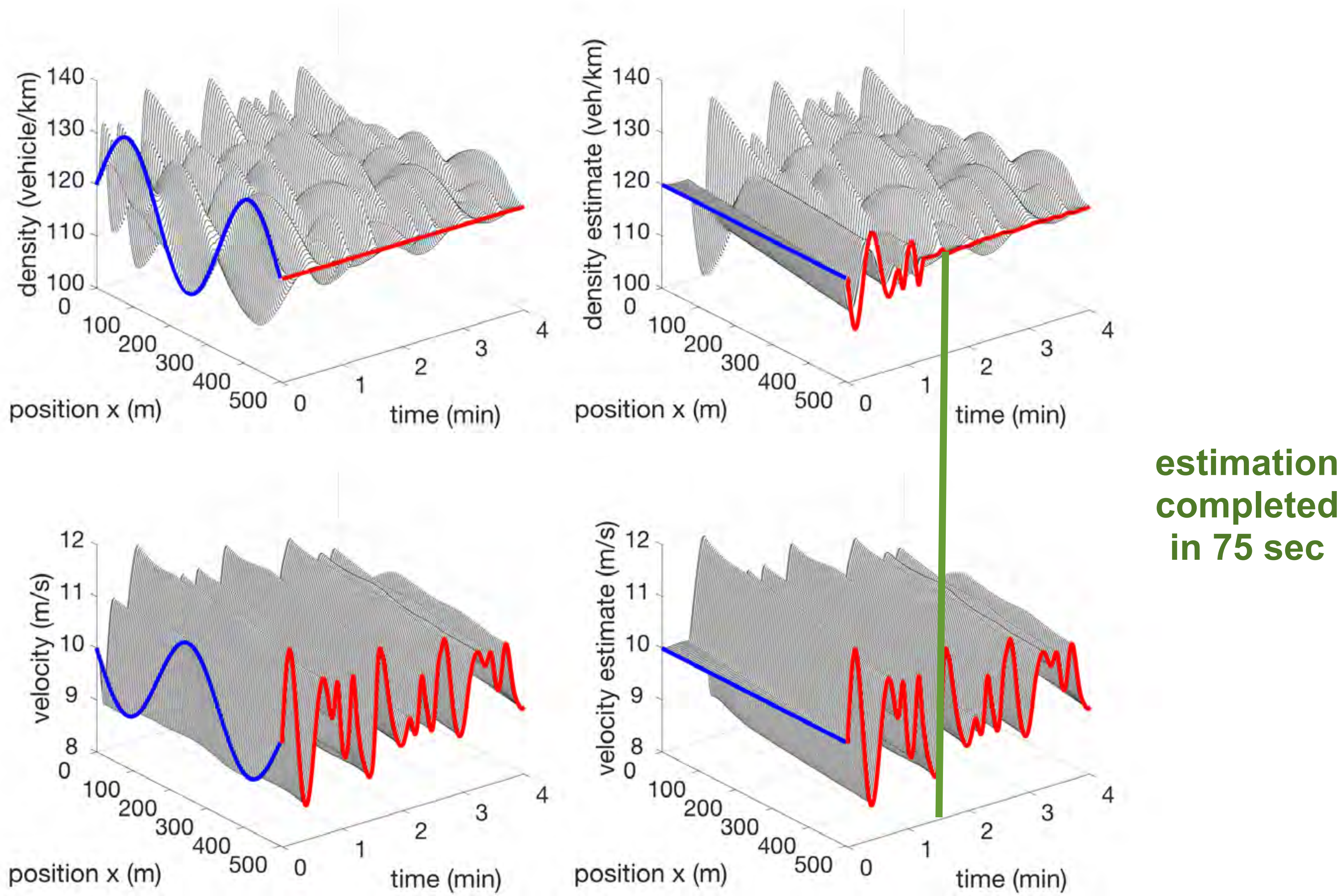
Boundary measurement (of velocity and flow):

$$Y(t) = \bar{w}(L, t)$$

Observer

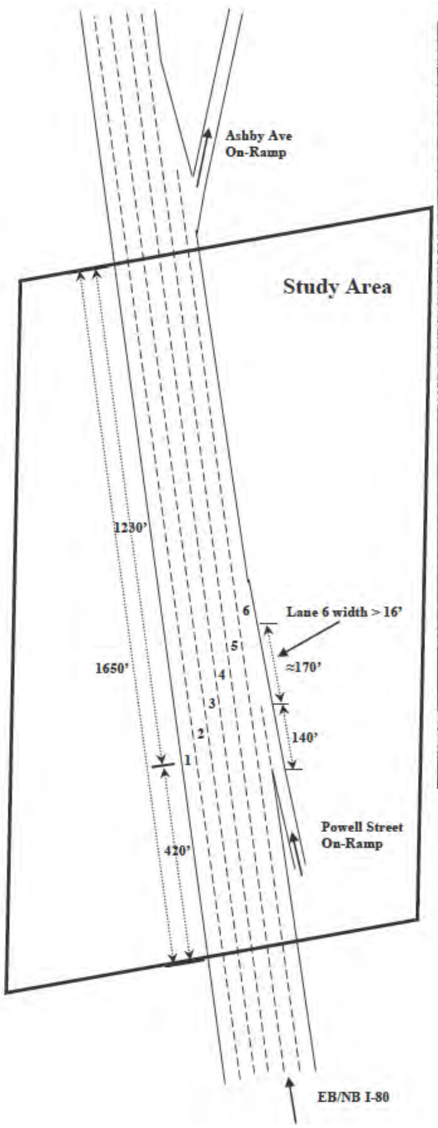
$$\begin{aligned}
\widehat{w}_t =& - v^\star \widehat{w}_x + r(x)(Y(t) - \widehat{w}(L,t)) \\
\widehat{v}_t =& (\gamma p^\star - v^\star)\widehat{v}_x + c(x)\widehat{w} + s(x)(Y(t) - \widehat{w}(L,t)) \\
\widehat{w}(0,t) =& - r_0 \widehat{v}(0,t) \\
\widehat{v}(L,t) =& r_1 Y(t) + U(t)
\end{aligned}$$

Density and velocity estimates



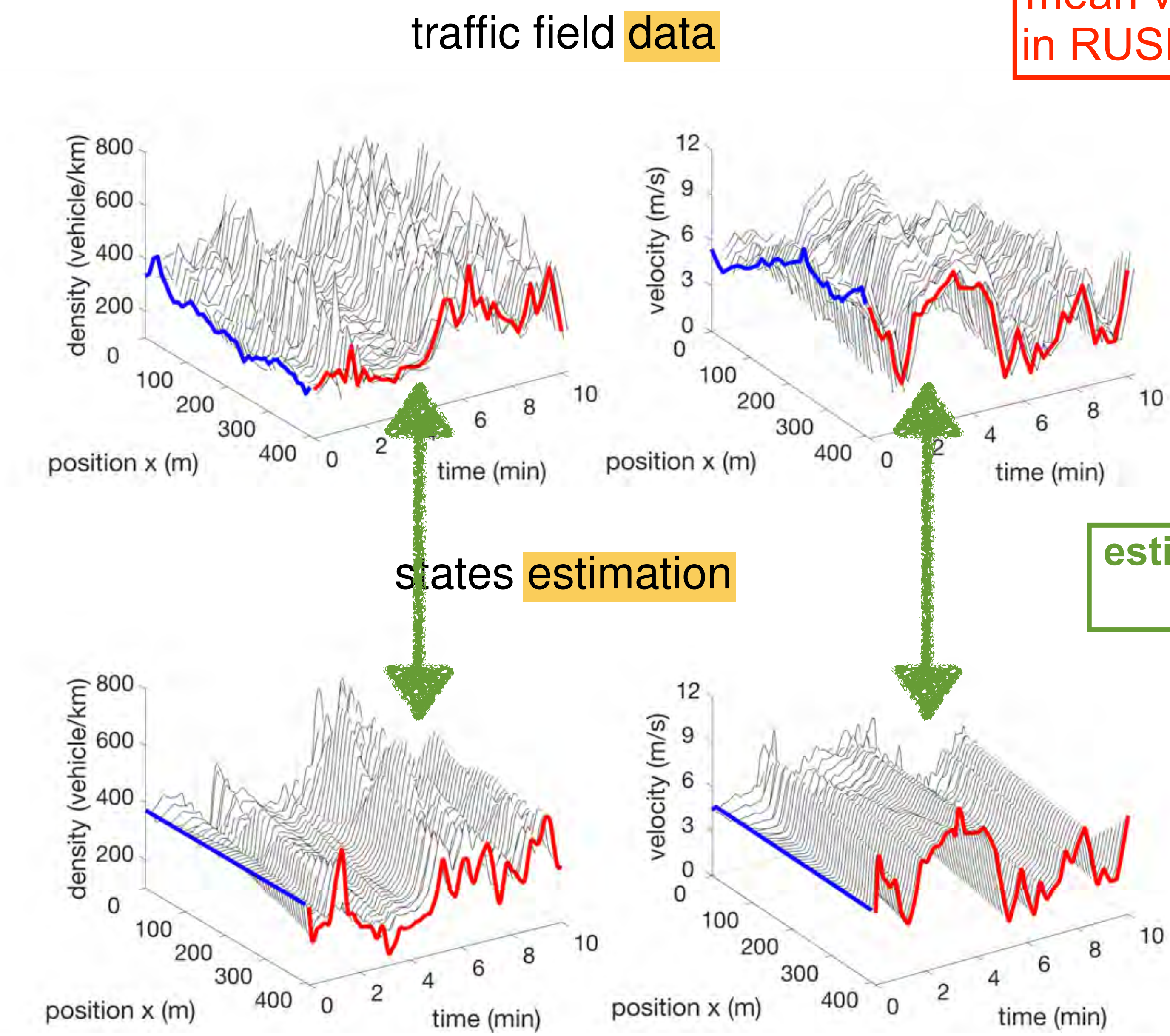
Data validation of boundary observer

Next Generation ~~Simulation~~ (NGSIM) traffic data I-80 in California



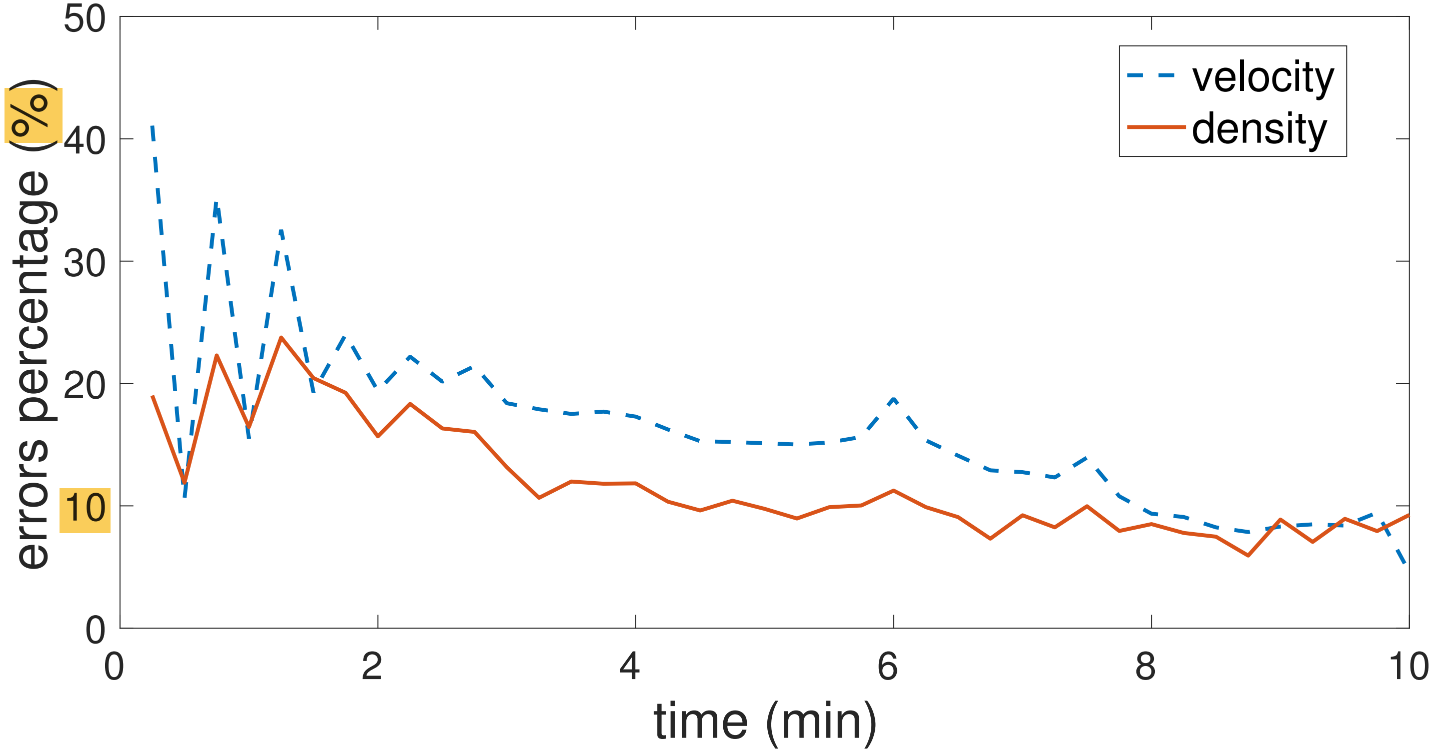
Traffic density and velocity for 5 : 15 pm - 5 : 30 pm

mean velocity 12 mph
in RUSH HOUR traffic



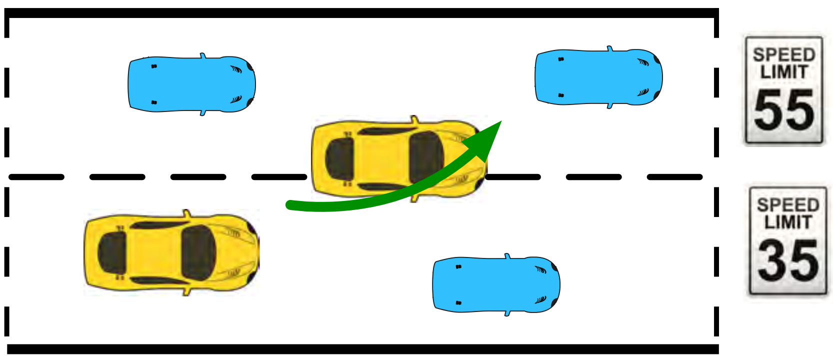
estimator converges
in 3-5 minutes

Estimation errors of boundary observer



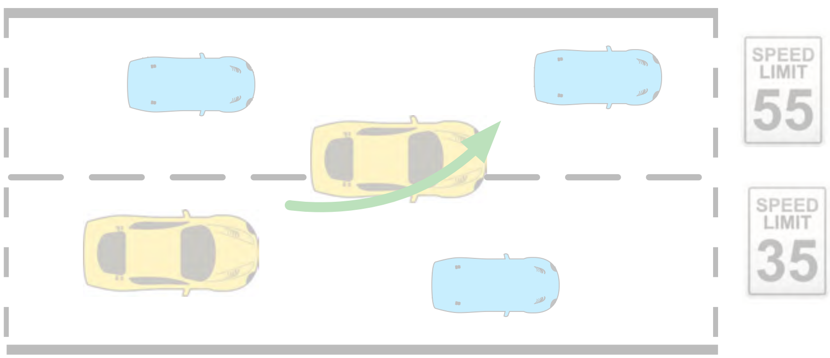
Two-lane and Two-class traffic congestion control

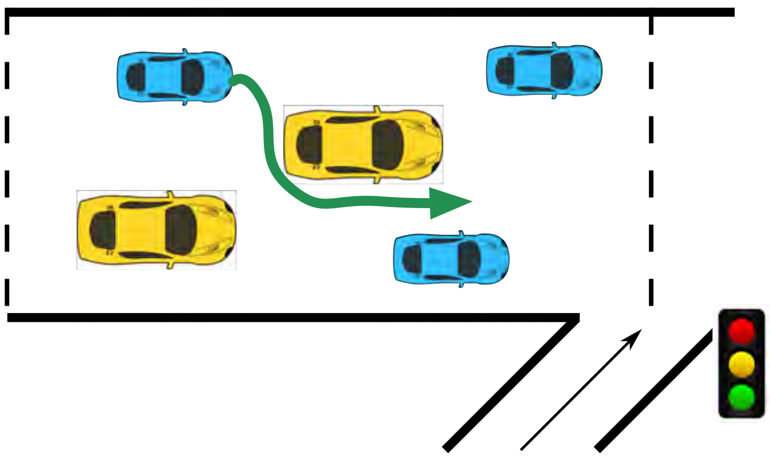
Two-lane and Two-class traffic congestion control



Lane changing segregates drivers into the more "risk-tolerant" ones in the **fast lane** and more **"risk-averse" ones** in the **slow lane**.

Two-lane and Two-class traffic congestion control





Lane changing segregates drivers into the more "risk-tolerant" ones in the **fast lane** and more "**risk-averse**" **ones** in the **slow lane**.

Mixed vehicle types induce **creeping effect** where the **slow and bulky** vehicles block the traffic and **small and fast** vehicles getting through.

Two-lane ARZ model

lane changes ~ heat exchanger

Coupled 4×4 nonlinear first-order hyperbolic PDEs

$$\partial_t \rho_f + \partial_x(\rho_f v_f) = \frac{1}{T_s}\rho_s - \frac{1}{T_f}\rho_f$$

$$\partial_t(\rho_f v_f) + \partial_x(\rho_f v_f^2) - (\rho_f \gamma p_f)\partial_x v_f = \frac{1}{T_s}\rho_s v_s - \frac{1}{T_f}\rho_f v_f + \frac{\rho_f(V(\rho_f) - v_f)}{T_f^e}$$

$$\partial_t \rho_s + \partial_x(\rho_s v_s) = \frac{1}{T_f}\rho_f - \frac{1}{T_s}\rho_s$$

$$\partial_t(\rho_s v_s) + \partial_x(\rho_s v_s^2) - (\rho_s \gamma p_s)\partial_x v_s = \frac{1}{T_f}\rho_f v_f - \frac{1}{T_s}\rho_s v_s + \frac{\rho_s(V(\rho_s) - v_s)}{T_s^e}$$

Two-class ARZ model

Coupled 4×4 nonlinear first-order hyperbolic PDEs

$$\partial_t \rho_1 + \partial_x(\rho_1 v_1) = 0$$

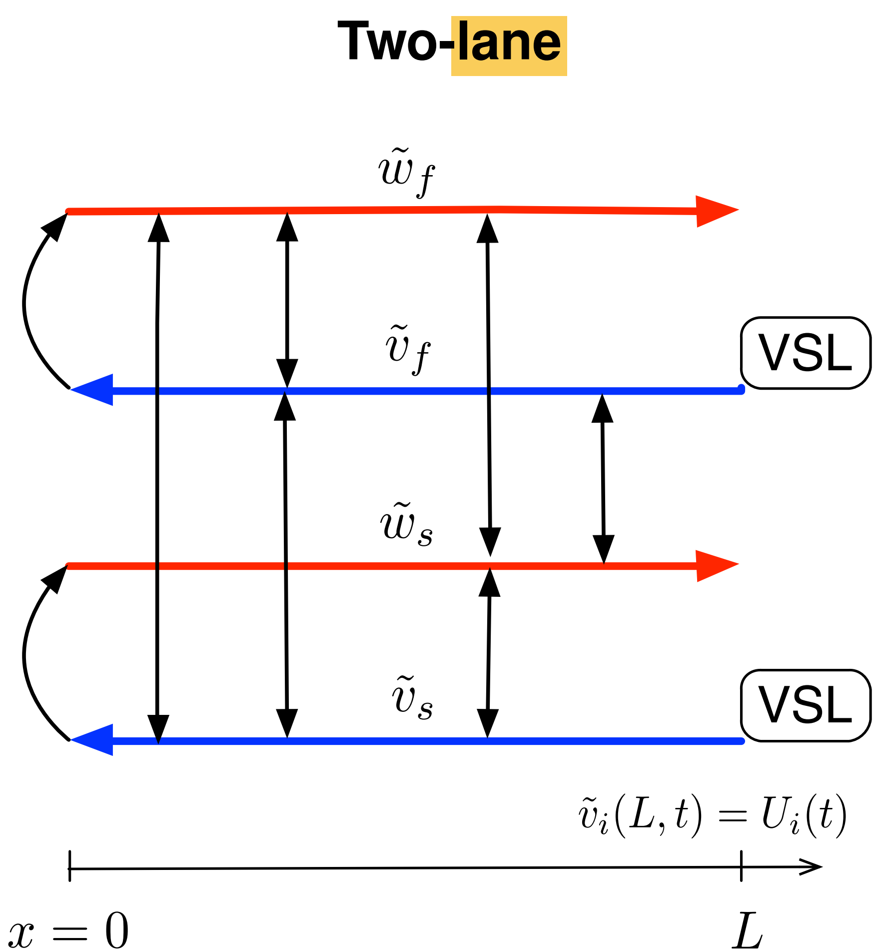
$$\partial_t(v_1 + p_1(AO)) + v_1 \partial_x(v_1 + p_1(AO)) = \frac{V_{e,1}(AO) - v_1}{\tau_1}$$

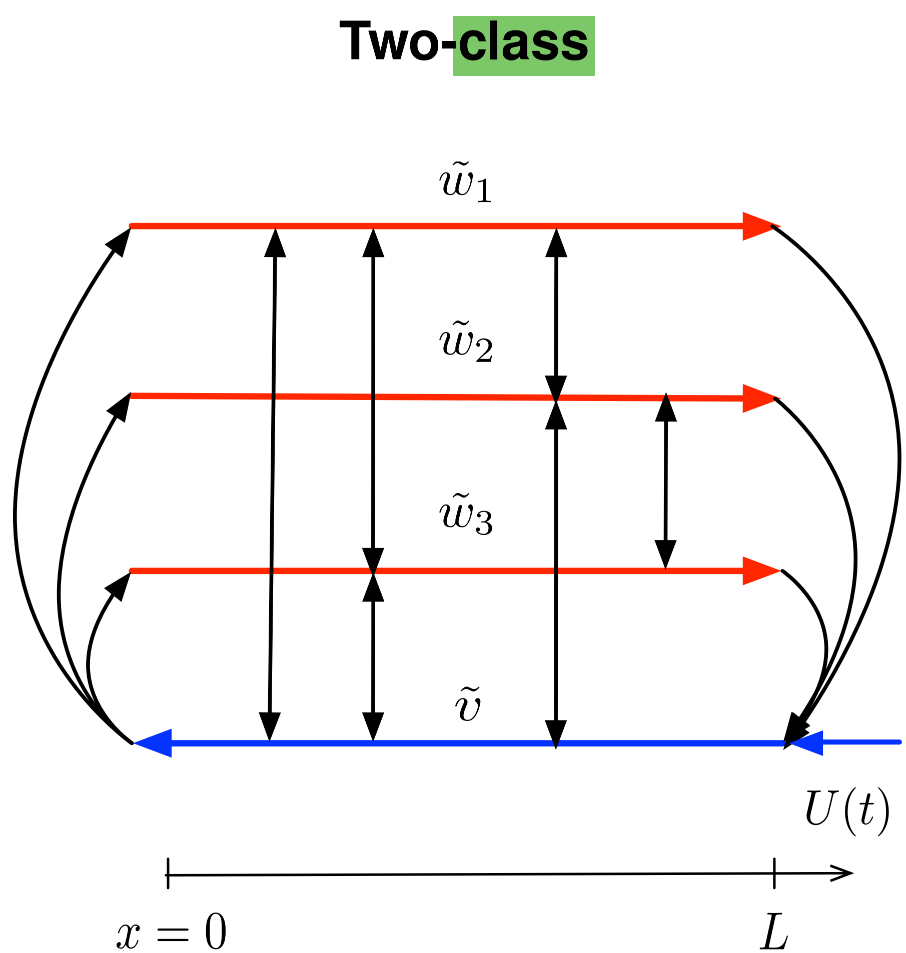
$$\partial_t \rho_2 + \partial_x(\rho_2 v_2) = 0$$

$$\partial_t(v_2 + p_2(AO)) + v_2 \partial_x(v_2 + p_2(AO)) = \frac{V_{e,2}(AO) - v_2}{\tau_2}$$

"Area Occupancy" $AO(\rho_1, \rho_2) = \frac{a_1 \rho_1(x,t) + a_2 \rho_2(x,t)}{W}$

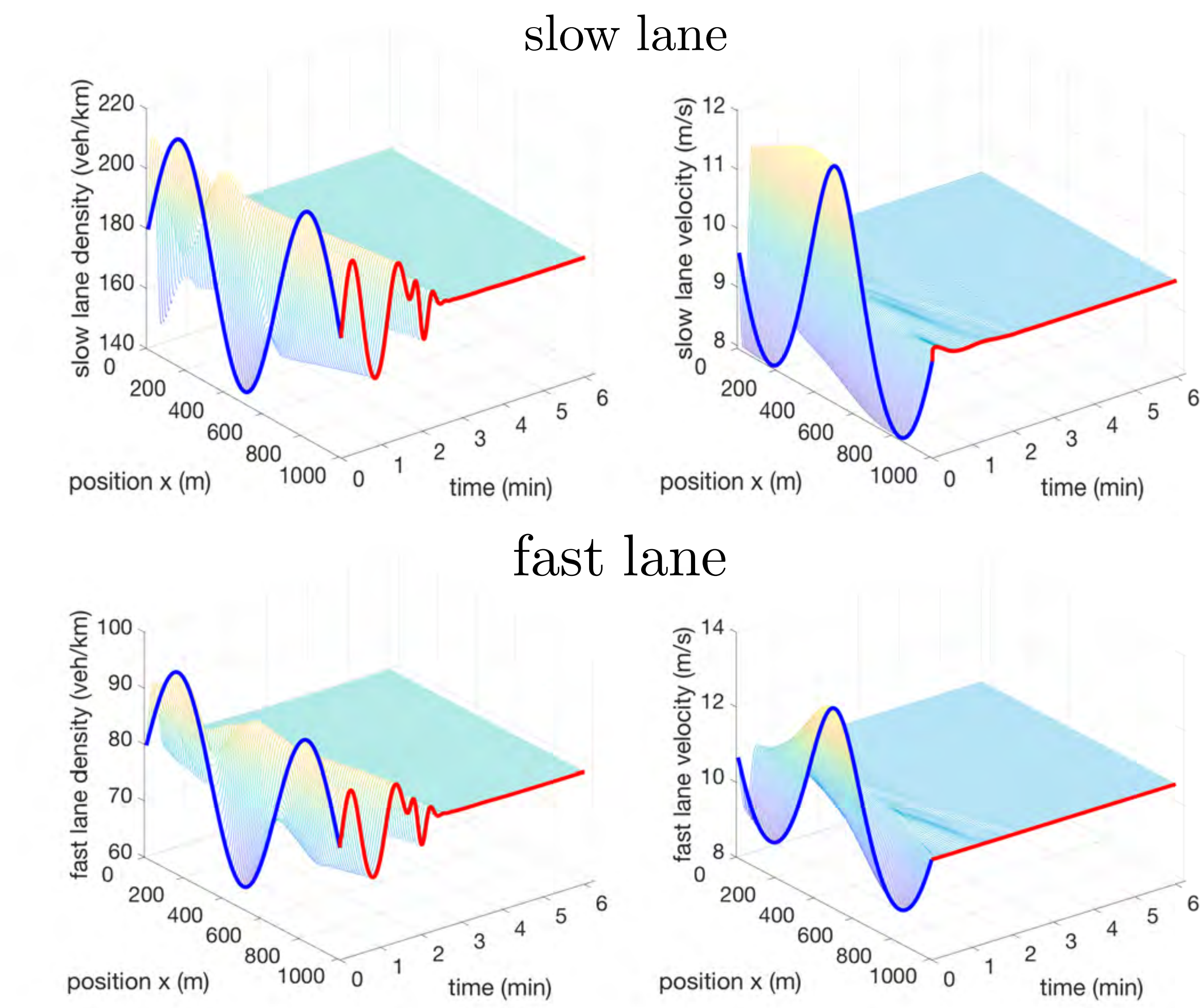
Coupled 4×4 nonlinear first-order hyperbolic PDEs



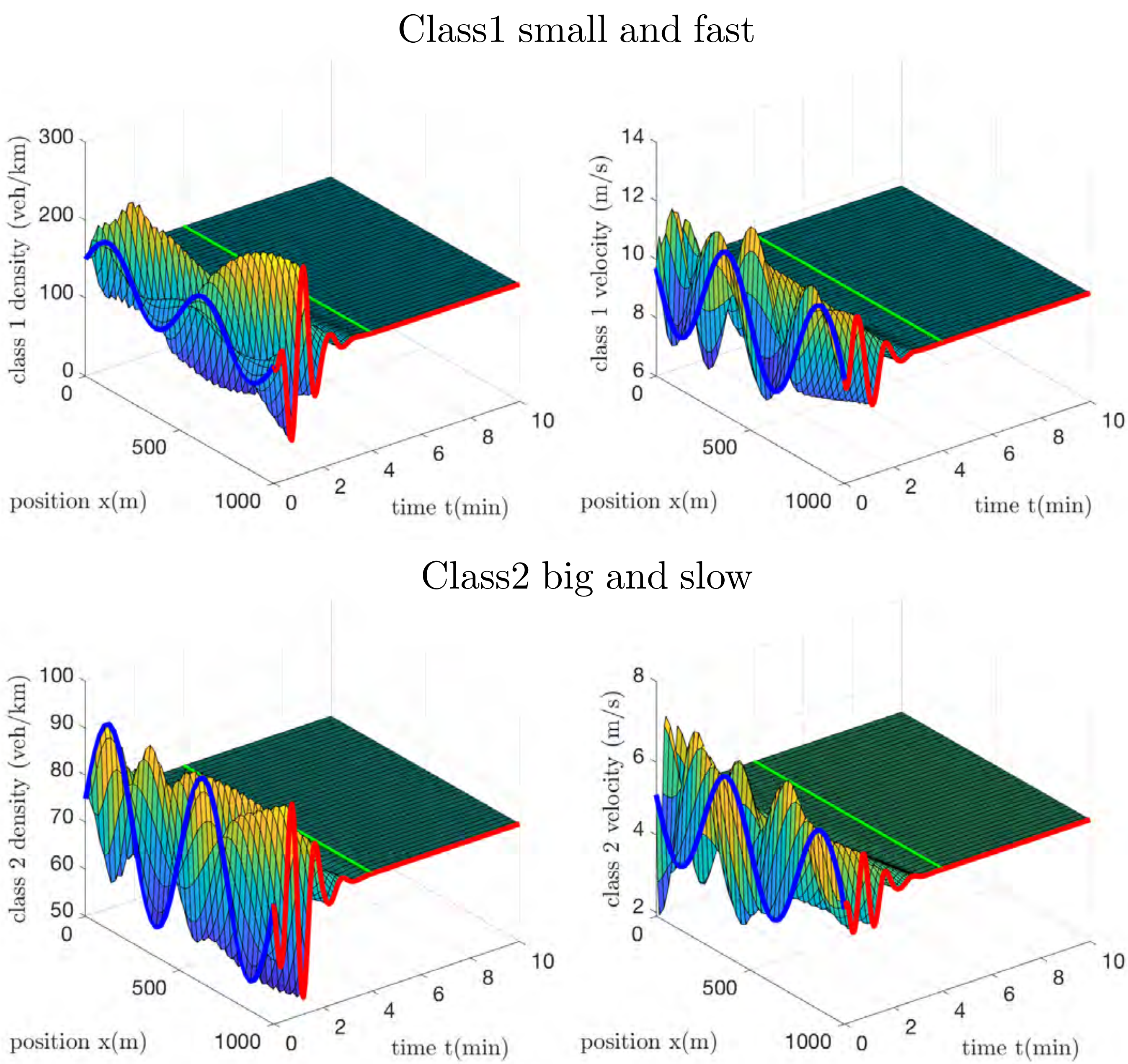


$2 + 2$ coupled PDE system **$3 + 1$ coupled PDE system**

Two-lane traffic full-state feedback simulation

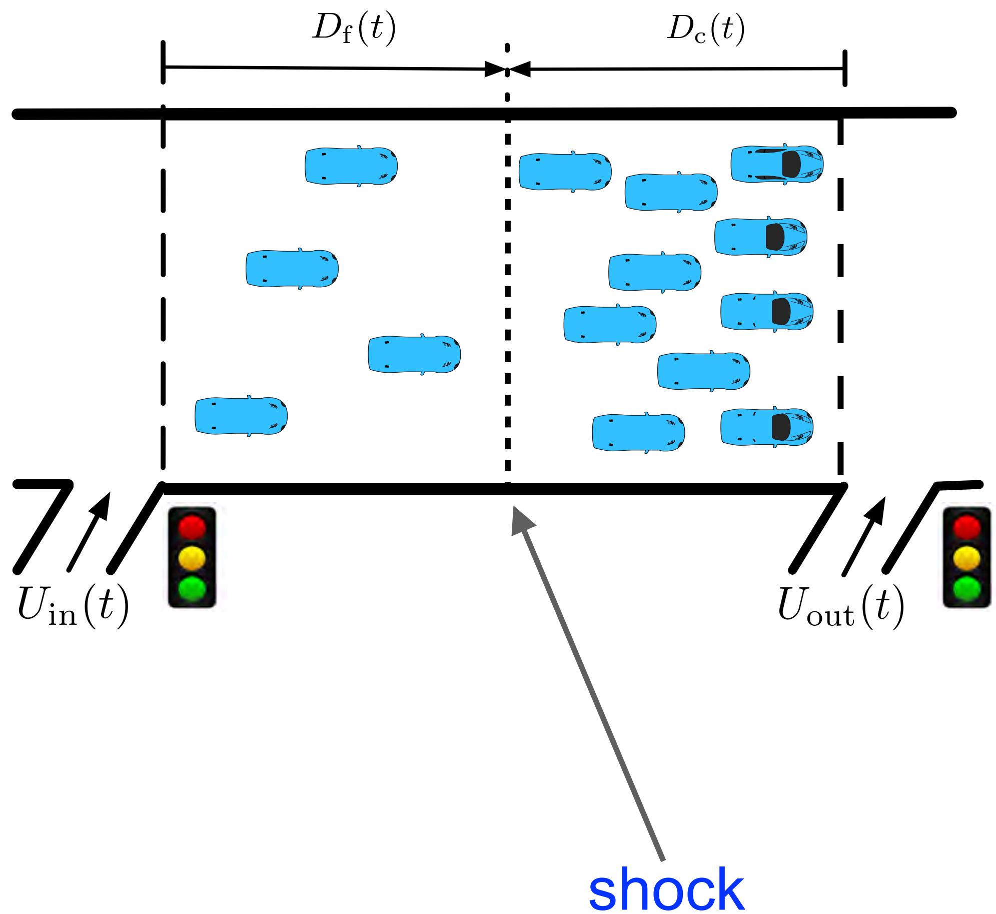


Two-class traffic full-state feedback simulation



Bilateral Boundary Control of

Moving Shockfront



$D_{\mathrm{f}}(t)$
$D_{\mathrm{c}}(t)$
$U_{\mathrm{in}}(t)$
$U_{\mathrm{out}}(t)$
shock

LWR traffic model

$$\frac{\partial}{\partial t}\rho_{\mathsf{f}} = -v_m \frac{\partial}{\partial x}\left(\rho_{\mathsf{f}} - \frac{\rho_{\mathsf{f}}^2}{\rho_m}\right)$$

density of "FREE traffic"
(upstream of shock)

LWR traffic model

$$\frac{\partial}{\partial t}\rho_{\mathsf{f}} = -v_m \frac{\partial}{\partial x}\left(\rho_{\mathsf{f}} - \frac{\rho_{\mathsf{f}}^2}{\rho_m}\right)$$

density of “FREE traffic”
(upstream of shock)

$$\frac{\partial}{\partial t}\rho_{\mathsf{C}} = -v_m \frac{\partial}{\partial x}\left(\rho_{\mathsf{C}} - \frac{\rho_{\mathsf{C}}^2}{\rho_m}\right)$$

density of “CONGESTED traffic”
(downstream of shock)

LWR traffic model

$$\frac{\partial}{\partial t}\rho_{\mathsf{f}} = -v_m \frac{\partial}{\partial x}\left(\rho_{\mathsf{f}} - \frac{\rho_{\mathsf{f}}^2}{\rho_m}\right)$$

density of "FREE traffic"
(upstream of shock)

$$\frac{\partial}{\partial t}\rho_{\mathsf{C}} = -v_m \frac{\partial}{\partial x}\left(\rho_{\mathsf{C}} - \frac{\rho_{\mathsf{C}}^2}{\rho_m}\right)$$

density of "CONGESTED traffic"
(downstream of shock)

$$\boxed{\frac{d}{dt}l(t) = v_m - \frac{v_m}{\rho_m}(\rho_{\mathsf{C}}(l(t),t) + \rho_{\mathsf{f}}(l(t),t))}$$

shock front location

(Rankine-Hugoniot jump condition)

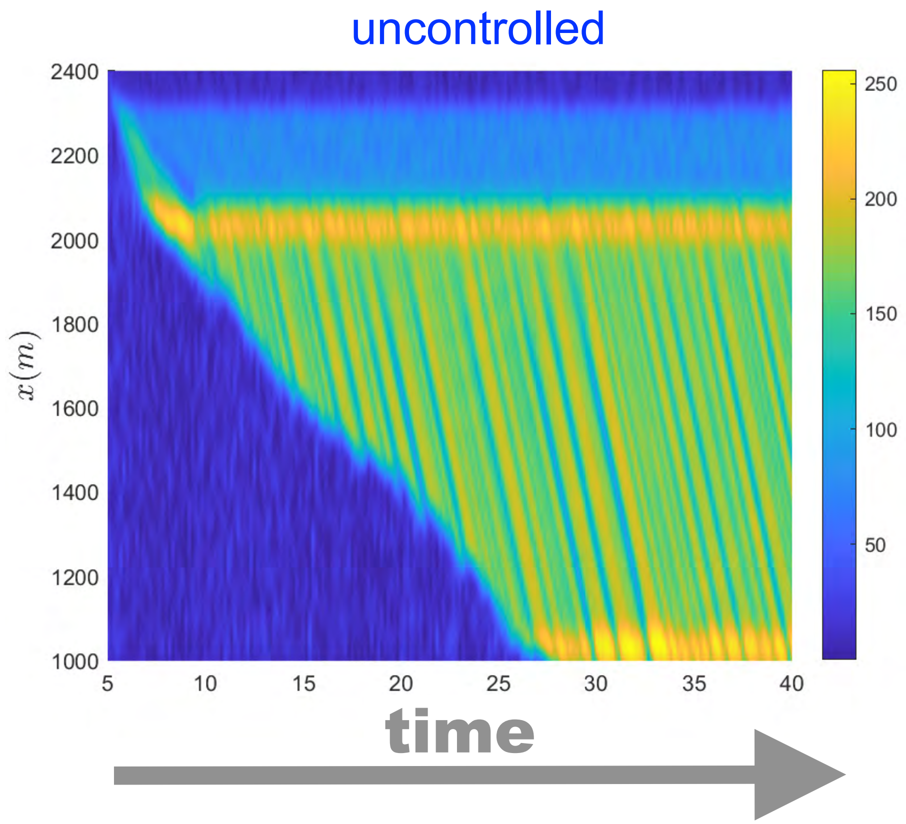
Bilateral predictor-based control

$$U_{\text{in}}(t) = K_{\text{f}} \left[X(t) - \frac{b}{u} \left(\int_0^{l(t)} \tilde{\rho}_{\text{f}}(\xi, t) d\xi + \int_{l(t)}^{\min\{L, 2l(t)\}} \tilde{\rho}_{\text{c}}(\xi, t) d\xi \right) \right]$$

$$U_{\text{out}}(t) = K_{\text{c}} \left[\boxed{X(t) - \frac{b}{u} \left(\int_{l(t)}^{L} \tilde{\rho}_{\text{c}}(\xi, t) d\xi + \int_{\max\{0, 2l(t) - L\}}^{l(t)} \tilde{\rho}_{\text{f}}(\xi, t) d\xi \right)} \right]$$

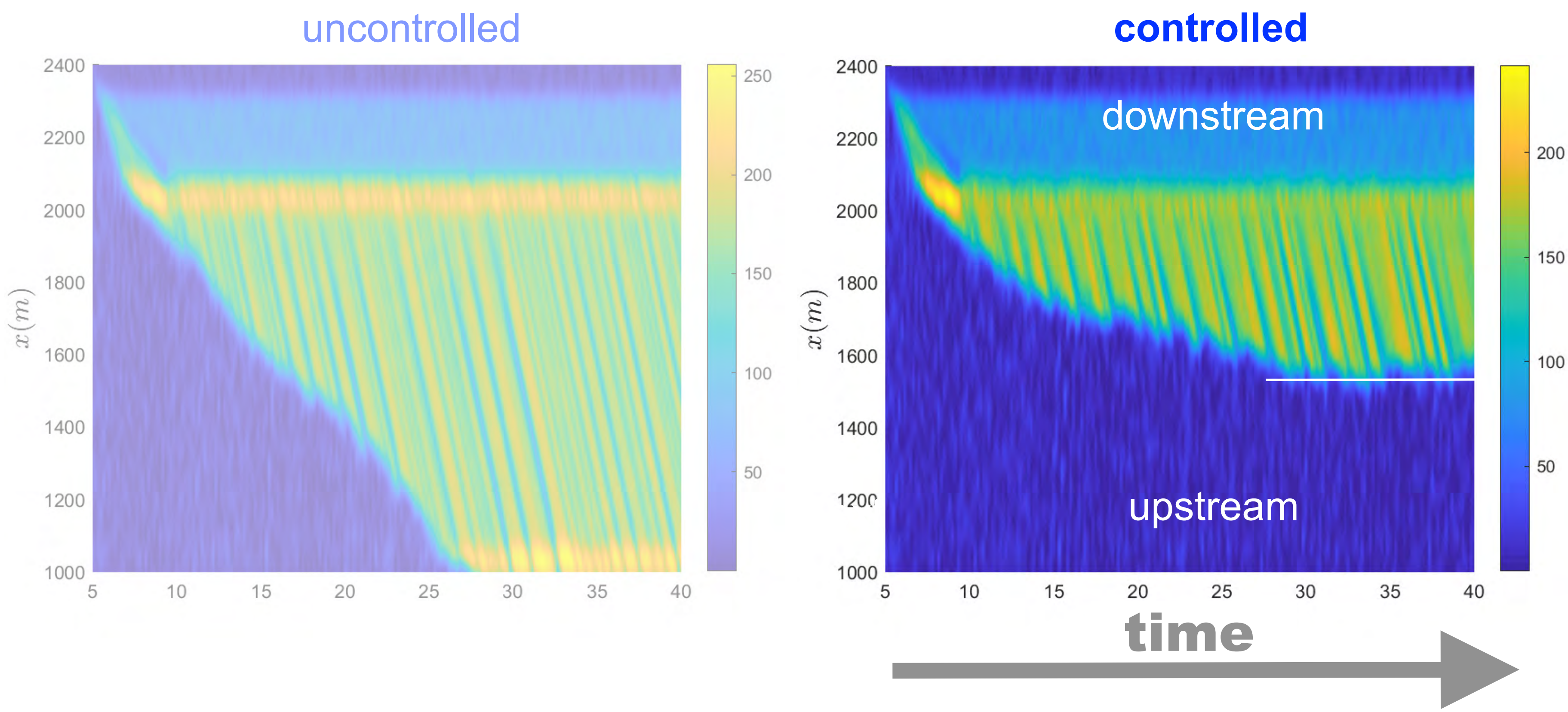
PREDICTION of shock position over the CONGESTED/downstream segment

Microscopic simulation experiment in AIMSUN



Open-loop becomes fully congested after 25 minutes

Microscopic simulation experiment in AIMSUN



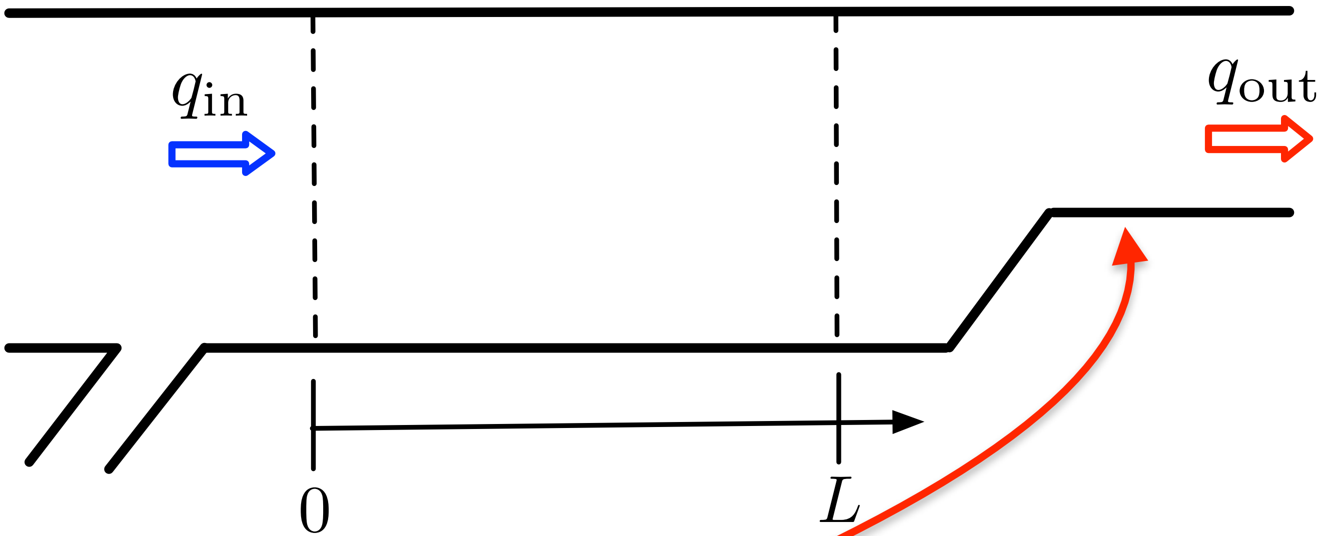
shockwave of

closed-loop stopped at 1600 m, leaving upstream traffic in free regime.

Extremum Seeking Control of

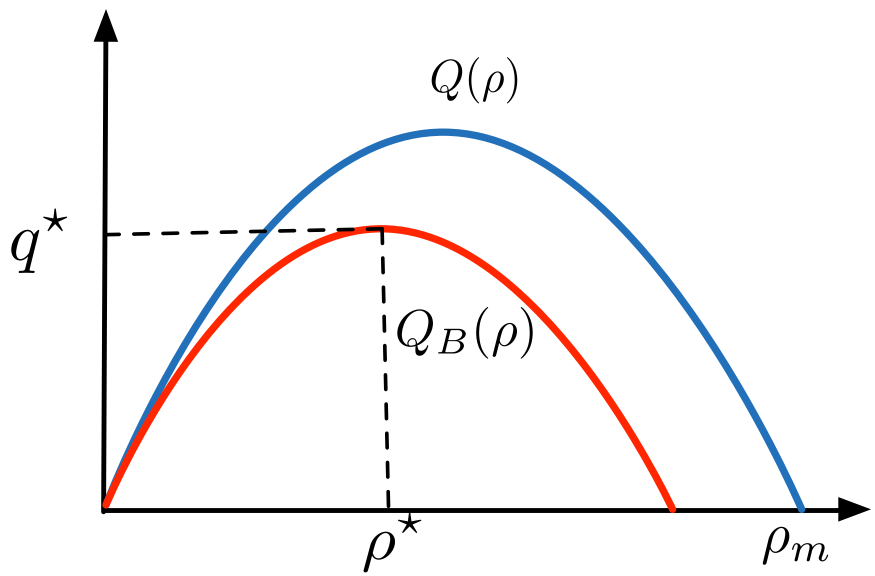
Downstream Traffic Bottleneck

Downstream traffic bottleneck



- **Capacity drop** at the bottleneck area, hard to model

Downstream traffic bottleneck



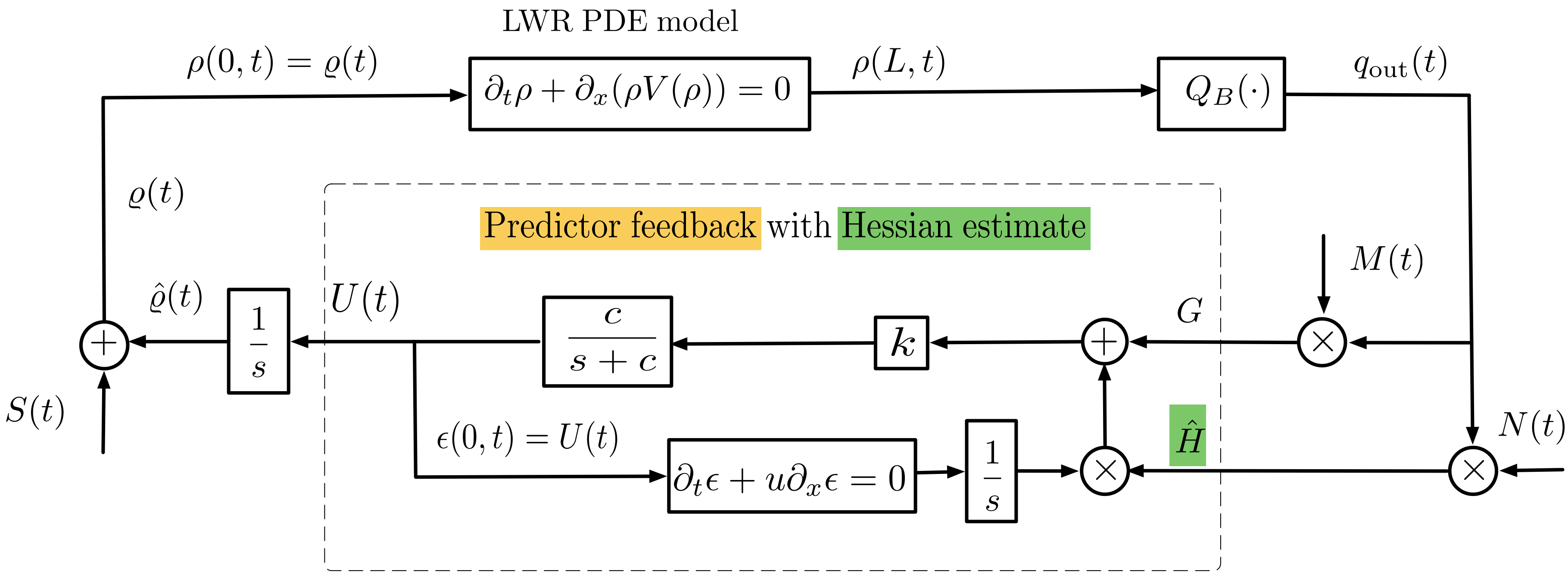
Upstream traffic dynamics with $q_{\text{in}}(t) = Q(\rho(0,t))$

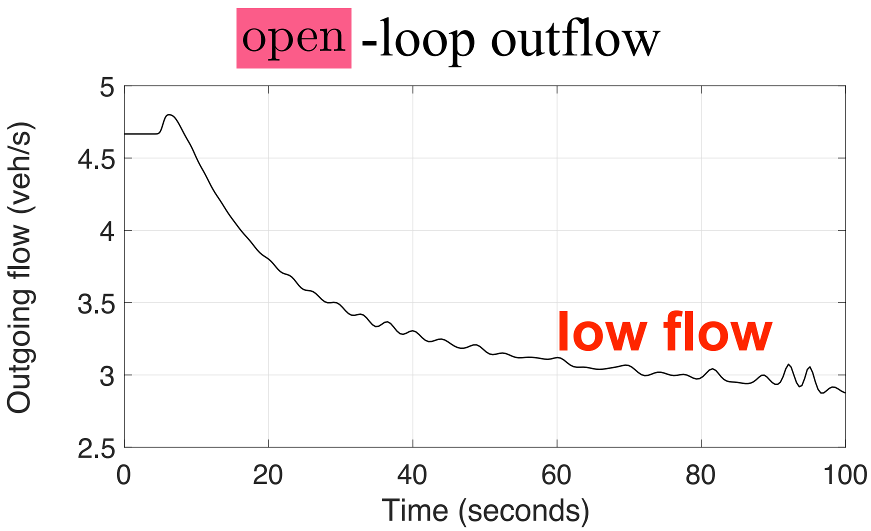
$$\partial_t \rho + \partial_x (Q(\rho)) = 0$$

Unknown quadratic map of bottleneck area

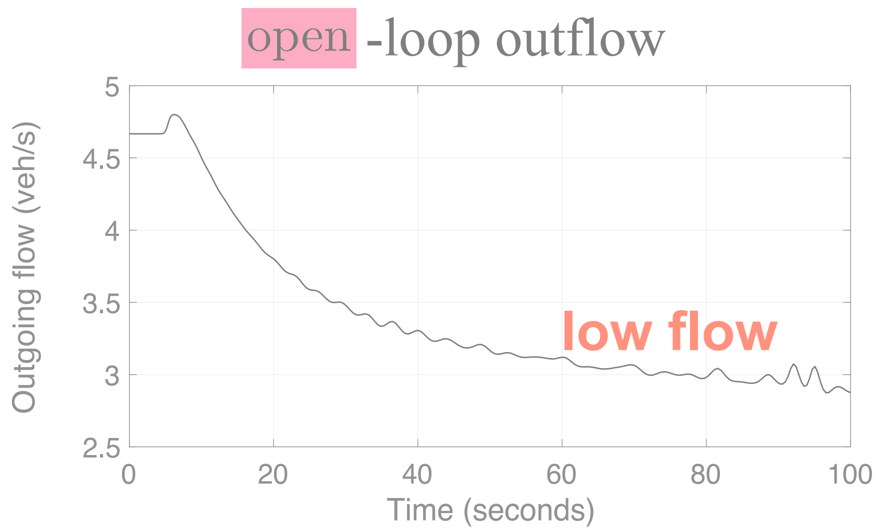
$$q_{\text{out}}(t) = Q_B(\rho(L,t)) = q^\star + \frac{H}{2}(\rho(L,t) - \rho^\star)^2$$

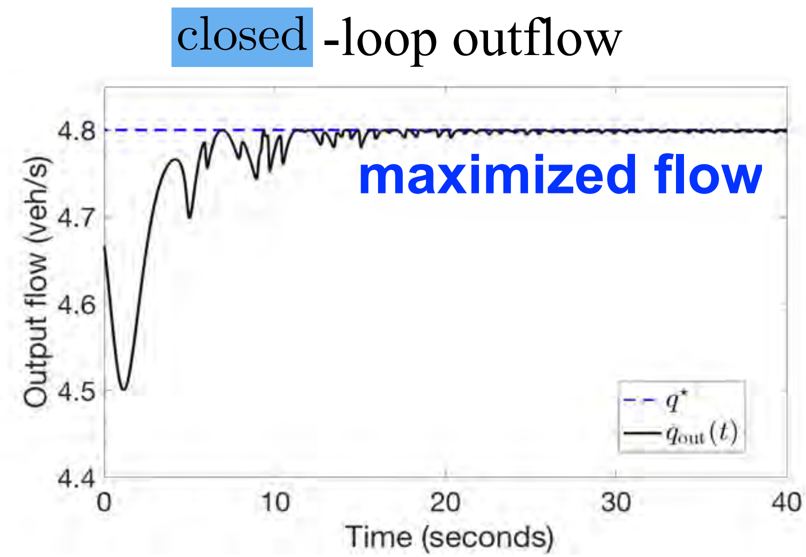
Extremum Seeking Control with delay





open -loop outflow
Outgoing flow (veh/s)
5
4.5
4
3.5
3
2.5
0
20
40
60
80
100
Time (seconds)
low flow


open -loop outflow
Outgoing flow (veh/s)
Time (seconds)
low flow


closed -loop outflow
Output flow (veh/s)
Time (seconds)
maximized flow
q^*
$q_{\mathrm{out}}(t)$

Thanks for your attention!